THE QUARTIC DISTRIBUTION AND ITS PROPERTIES

by

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ABSTRACT

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We will introduce a two-parameter probability distribution, the Quartic, in this paper. The distribution has the interesting property of having lighter tails than the normal, while still retaining many of its other favorable characteristics. We will begin by establishing many of the basic properties of the Quartic, such as symmetry, the integrating constant, cumulative probably distribution, moments, and so forth. We will then proceed onto developing estimators for its parameters, using both the maximum likelihood as well as the method of moments techniques.

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1 INTRODUCTION

In this paper, a probability distribution called the Quartic shall be investigated. Basic properties of the distribution shall be developed and verified, such as its validity as a density function, moments, cumulative density function, etc. After looking at the mathematical proprieties of the distribution, we shall turn our focus to estimating the parameters of the distribution with two techniques: maximum likelihood estimation and the method of moments. We hope this paper and the research within it shall help lay the groundwork for further exploration of the quartic distribution.

There are a few references to a quartic distribution in the current body of research. However, both are very different than the approach being taken in this thesis. First, there is the quartic exponentiated distribution, which is defined as follows:

$$f(X) = \frac{1}{N}\exp(-(a_1x + a_2x^2 + a_3x^3 + a_4x^4))(\text{see } [1]).$$

The quartic distribution in this paper is actually a particular case of this distribution. Second, there is another distribution called the The complex Bingham quartic distribution. This distribution is an extension of a distribution called the complex Bingham distribution in order to give it more use in wide application(See [2]). Again, this distribution is quite different from the distribution defined in this paper.

2 PROPERTIES OF THE QUARTIC DISTRIBUTION

2.1 DEFINITION

A random variable, X, is distributed according to the Quartic distribution if its probability density function, pdf, is of the form:

$$f(X|\mu,\sigma) = \frac{\sqrt{2}}{\sigma\Gamma(\frac{1}{4})}\exp(-\frac{(x-\mu)^4}{4\sigma^4}),\ x \in (-\infty,\infty)$$

where μ is the population mean, and σ is the population kurtosis. To denote that X follows a Quartic distribution with parameters μ and σ, we say $X \sim Q(\mu, \sigma)$.

We observe that the pdf of the Quartic is very similar to that of the normal (i.e. $f(X|\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp(-\frac{(x-\mu)^2}{2\sigma^2})$), but the exponent is raised to the fourth power as opposed to the second power in the normal. Next, the normalizing constant for the Quartic is also slightly different than that of the normal. Furthermore, if we were to take $a_4 = \dfrac{1}{4\sigma^4}$, $a_3 = -\dfrac{\mu}{\sigma^4}$, $a_2 = \dfrac{3\mu^2}{2\sigma^4}$, $a_1 = -\dfrac{\mu^3}{\sigma^4}$, we can see that the quartic is a particular case of the quartic exponentiated distribution.

On the following page is a graph of the quartic distrusting superimposed on the normal. The reader can see the effect of raising the exponentiated term, $\dfrac{x-\mu}{\sigma}$, to the fourth power as opposed to the second. The tails of the quartic are much narrower and the top is much flatter, or is more negatively kurtic, than the normal. Yet, the quartic still retains the symmetry and bell-like shape of the normal.

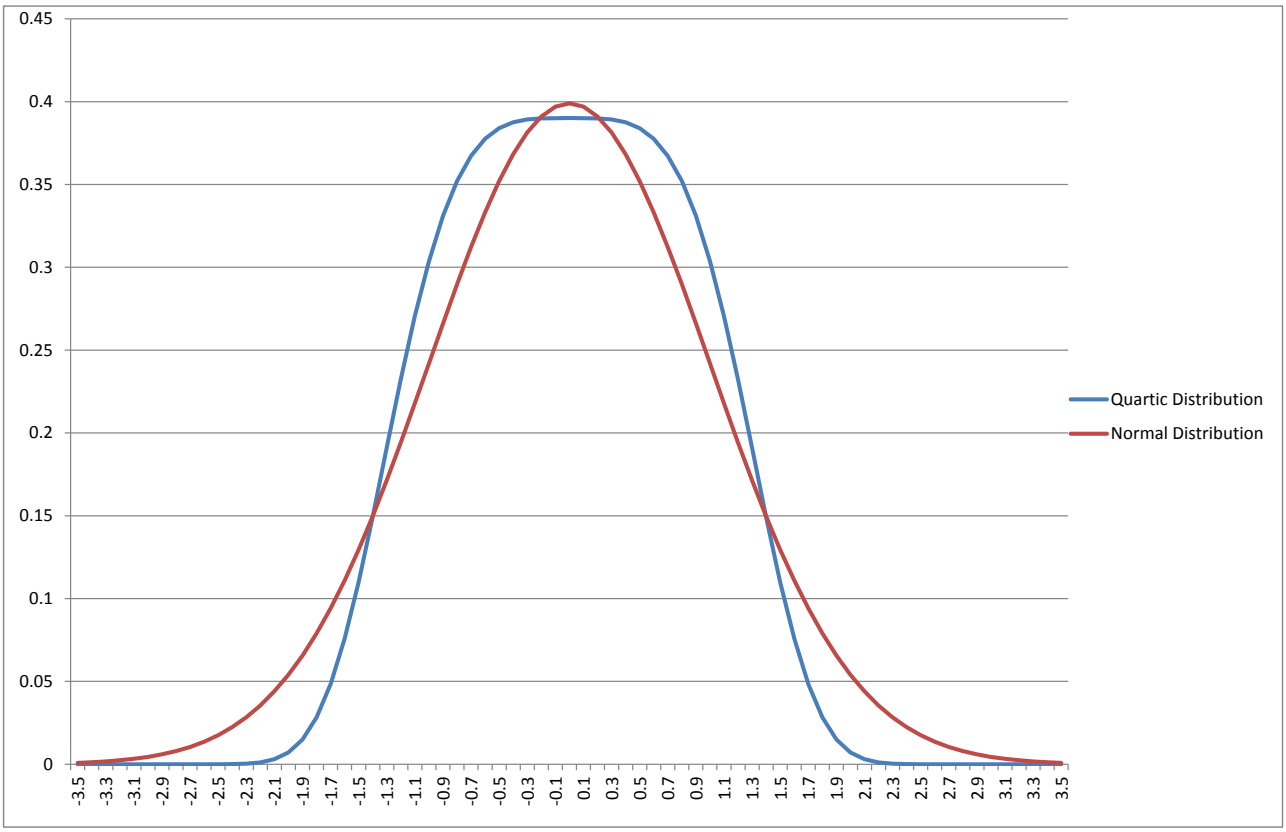


Figure 1: The standard pdf of the quartic and the normal. Note the narrower tails and flatter top of the quartic.

2.2 THE STANDARD QUARTIC

Similar to the standard normal in the normal distribution, the "Standard Quartic" can also be defined similarity. The Standard Quartic is the situation when $\mu = 0$ and $\sigma = 1$. Its pdf is as follows:

$$f(x|\mu = 0, \sigma = 1) = \frac{\sqrt{2}}{\Gamma(\frac{1}{4})} \exp\left(-\frac{x^4}{4}\right)$$

2.3 DENSITY FUNCTION

First, we need to prove that the Quartic is, in fact, a valid probability density function. This means two things:

$$1) f(x) \geq 0, (\forall x \in \mathbf{R})$$
$$2) \int_{-\infty}^{+\infty} f(x)\, dx \ = 1$$

The proof for 1) is straightforward, since the function, since the function in 1) is always positive.

Before we prove the integral over the real line equals one, we must first prove that the distribution is symmetric about the mean μ, as this property will prove essential in finding a closed form solution to the integral over the real line. The proof for symmetry is as follows:

$$\begin{aligned} f(\mu + t) &= \frac{\sqrt{2}}{\Gamma(1/4)\sigma} \exp\left[-\frac{1}{4}\left(\frac{\mu + t - \mu}{\sigma}\right)^4\right] \\ &= \frac{\sqrt{2}}{\Gamma(1/4)\sigma} \exp\left[-\frac{1}{4}\left(\frac{t}{\sigma}\right)^4\right] \\ &= \frac{\sqrt{2}}{\Gamma(1/4)\sigma} \exp\left[-\frac{1}{4}\left(\frac{-t}{\sigma}\right)^4\right] \\ &= \frac{\sqrt{2}}{\Gamma(1/4)\sigma} \exp\left[-\frac{1}{4}\left(\frac{(\mu - t) - \mu}{\sigma}\right)^4\right] \\ &= f(\mu - t)\,. \end{aligned}$$

Thus, the pdf is symmetric about the mean, μ.

Now we can show the proof to equation 2). Making the substitution: $z = \dfrac{(x - \mu)}{\sigma}$, we obtain:

Proof.

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)dx &= \int_{-\infty}^{\infty} \frac{\sqrt{2}}{\Gamma(1/4)} \exp\left[-\frac{1}{4}z^4\right] dz \\ &= 2\int_{0}^{\infty} \frac{\sqrt{2}}{\Gamma(1/4)} \exp\left[-\frac{1}{4}z^4\right] dz, \text{by symmetry}\end{aligned}$$

Substituting $u = \frac{1}{4}z^4 \Leftrightarrow x = 4u^{\frac{1}{4}}$, we obtain: $du = z^3dz \Leftrightarrow dz = (4u)^{\frac{-3}{4}}du$. Hence,

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)dx &= \frac{2\sqrt{2}}{\Gamma(1/4)} \int_{0}^{\infty} \exp[-u]4u^{-\frac{3}{4}}du \\ &= \frac{2\sqrt{2}}{\Gamma(1/4)} 4^{-\frac{3}{4}} \int_{0}^{\infty} \exp[-u]u^{\frac{1}{4}-1}du \\ &= \frac{1}{\Gamma(1/4)}\Gamma(1/4) \\ &= 1\end{aligned}$$

□

Thus, the pdf is valid.

Next, we shall differentiate the standard quartic density function in order to find its critical values.

$$f'(x) = -\frac{\sqrt{2}}{\Gamma(1/4)} \exp\left\{-\frac{1}{4}x^4\right\} x^3.$$

By the first derivative test, $f(x)$ is increasing while x is positive, and decreasing while x is negative.

Next, we can use the second derivative to compute the points on inflection:

$$\begin{aligned}f''(x) &= \frac{\sqrt{2}}{\Gamma(1/4)} \left\{\exp\left[-\frac{1}{4}x^4\right]\left(-3x^2 + x^6\right)\right\} \\ &= \frac{\sqrt{2}}{\Gamma(1/4)} x^2 \exp\left[-\frac{1}{4}x^4\right]\left(-3 + x^4\right).\end{aligned}$$

As we can see, the second derivative is zero at $x = 0$ and $|x| = 3^{\frac{1}{4}}$ Thus, the points of

inflection are $x^4 = 3$. So, we can see that the inflection points are: $x = \pm 3^{\frac{1}{4}}$.

The graph of $f(x)$ is concave downward provided $|x| > 3^{1/4}$ and concave up when $|x| < 3^{1/4}$. Note that the points of inflection are slightly further out than that of the normal, whose inflection points are ±1 standard deviation from the mean. This helps explain how flat this distribution is with respect to the normal.

2.4 MOMENTS

Due to the complexity of the pdf, a closed-form expression for the quartic's moment generating function is intractable. However, if we were to take the **standard quartic**, and solve $E(X^m)$, then we could obtain closed form solutions for the moments.
Since the pdf is symmetric about the mean, thus all odd moments vanish, we can further simplify this expectation to $E(X^{2m})$.

Theorem 1. *All even moments of the Quartic distribution can be given by:*

$$E\left(X^{2m}\right) = \frac{2^m}{\Gamma(\frac{1}{4})}\Gamma\left(\frac{2m+1}{4}\right), \forall m \in \boldsymbol{N}$$

Proof.

$$\begin{aligned} E\left(X^{2m}\right) &= 2\int_0^\infty x^{2m} f(x)dx \\ &= \frac{2\sqrt{2}}{\Gamma(\frac{1}{4})}\int_0^\infty x^{2m}\exp\left(-\frac{x^4}{4}\right)dx \end{aligned}$$

To simplify the integral, let $u = \frac{1}{4}x^4 \Rightarrow du = x^3 dx$ so that:

$$\begin{aligned} E\left(X^{2m}\right) &= \frac{2\sqrt{2}}{\Gamma(\frac{1}{4})}\int_0^\infty (4u)^{m/2}\exp\left(-u\right)(4u)^{-3/4}du \\ &= \frac{2\sqrt{2}}{\Gamma(\frac{1}{4})}\int_0^\infty (4u)^{(2m-3)/4}\exp\left(-u\right)du \\ &= \frac{2^m}{\Gamma(\frac{1}{4})}\int_0^\infty u^{(2m-3)/4}\exp\left(-u\right)du \\ &= \frac{2^m}{\Gamma(\frac{1}{4})}\Gamma\left(\frac{2m+1}{4}\right) \end{aligned}$$

□

We now have a closed form solution for the central moments about the mean of this distribution.

2.4.1 CENTRAL STANDARD MOMENTS

If we define $Q = \frac{X-\mu}{\sigma}$, solve for X, take both sides to $2m^{th}$ power, and take the expectation, then we can have a more general case of the moments of this distribution.

Theorem 2. *The even central standard moments for the Quartic distribution are of the following form:*

$$E(X^{2m}) = \sum_{i=1}^{n}\binom{n}{i}\mu^{n-i}E\left(Q^i\right)\sigma^i$$

.

Proof.

$$\begin{aligned} Q &= \frac{X-\mu}{\sigma} \\ &\Rightarrow X = \mu + Q\sigma \\ &\Rightarrow X^{2m} = (\mu + Q\sigma)^{2m} \\ &\Rightarrow E(X^{2m}) = E((\mu + Q\sigma)^{2m}) \end{aligned}$$

We can expand this using the Binomial Theorem.

$$E(X^{2m}) = \sum_{i=1}^{2m} \binom{2m}{i} \mu^{n-i} E\left(Q^i\right) \sigma^i$$

□

This result will be important in a coming section.

2.4.2 MACLAURIN SERIES EXPANSION OF THE MOMENT GENERATING FUNCTION

One method of finding the moment generating function of the Quartic is to estimate it using a MacLaurin Series expansion.

Theorem 3. *The moment generating function for the standard Quartic distribution can be expressed with the follow summation:*

$$M_X(t) = \sum_{m=0}^{\infty} \frac{t^{2m} 2^m \Gamma(\frac{2m+1}{4})}{(2m)! \Gamma(\frac{1}{4})}$$

Proof.

$$\begin{aligned} M_X(t) &= E[\exp[tX]] \\ &= E\left[\sum_{n=0}^{\infty} \frac{(tx)^n}{n!}\right], \text{by the MacLaurin Series expansion for } e^{tx} \\ &= \sum_{n=0}^{\infty} \frac{t^n E[X^n]}{n!} \\ &= \sum_{m=0}^{\infty} \frac{t^{2m} 2^{2m} \Gamma(\frac{2m+1}{4})}{(2m)! \Gamma(\frac{1}{4})}, \text{By the results in Section 2.4} \end{aligned}$$

Now, we must show the series converges. We can achieve this with the ratio test.

$$
\begin{aligned}
a_m &= \frac{t^{2m}2^m\Gamma(\frac{2m+1}{4})}{(2m)!\Gamma(\frac{1}{4})} \\
a_{m+1} &= \frac{t^{2(m+1)}2^{2(m+1)}\Gamma(\frac{2(m+1)+1}{4})}{(2(m+1))!\Gamma(\frac{1}{4})} = \frac{t^{2m+2}2^{2m+2}\Gamma(\frac{2m+3}{4})}{(2m+2)!\Gamma(\frac{1}{4})} \\
\therefore \left|\frac{a_{m+1}}{a_m}\right| &= \frac{t^{2m+2}2^{2m+2}\Gamma(\frac{2m+3}{4})}{(2m+2)!\Gamma(\frac{1}{4})} \times \frac{(2m)!\Gamma(\frac{1}{4})}{t^{2m}2^m\Gamma(\frac{2m+1}{4})} = \frac{4x^2}{(2m+2)(2m+1)} \times \frac{\Gamma(\frac{2m+3}{4})}{\Gamma(\frac{2m+1}{4})} \\
&\leq \frac{4x^2}{(2m+2)(2m+1)} \times \frac{\Gamma(\frac{2m+5}{4})}{\Gamma(\frac{2m+1}{4})} = \frac{4x^2(\frac{2m+1}{4})\Gamma(\frac{2m+1}{4})}{(2m+2)(2m+1)\Gamma(\frac{2m+1}{4})} \\
&= \frac{x^2}{2m+2} \to 0 \text{ as } m \to \infty \\
\therefore \quad &\left|\frac{a_{m+1}}{a_m}\right| \to L = 0 \text{ as } m \to \infty, \text{by the Squeeze Theorem} \\
\because \quad & L = 0 \leq 1, \sum_{m=0}^{\infty} a_m \text{ converges absolutely by the Ratio Test}
\end{aligned}
$$

□

2.5 THE STANDARD QUARTIC CDF

Like any density function, we can develop the quartic cumulative density function.

$$
\begin{aligned}
F(x|\mu,\sigma) &= \int_{-\infty}^{q} \frac{\sqrt{2}}{\Gamma(\frac{1}{4})} \exp\left(-\frac{t^4}{4}\right) dt \\
&= \frac{1}{2} + \frac{\sqrt{2}}{\Gamma(\frac{1}{4})} \text{sign}(q) \int_0^{|q|} \exp\left(-\frac{t^4}{4}\right) dt \\
&= \frac{1}{2} + \frac{\sqrt{2}}{\Gamma(\frac{1}{4})} \text{sign}(q) \int_0^{\frac{1}{4}q^4} \exp(-u)\,(4u)^{-\frac{3}{4}} du \\
&= \frac{1}{2} + \frac{\sqrt{2}}{\Gamma(\frac{1}{4})} \frac{1}{2\sqrt{2}} \text{sign}(q) \int_0^{\frac{1}{4}q^4} \exp(-u)\,(u)^{-\frac{3}{4}} du \\
&= \frac{1}{2} + \frac{1}{2}\frac{1}{\Gamma(\frac{1}{4})} \text{sign}(q) \int_0^{\frac{1}{4}q^4} \exp(-u)\,(u)^{\frac{1}{4}-1} du \\
&= \frac{1}{2} + \frac{1}{2} \text{sign}(q) \frac{\gamma(\frac{1}{4}, \frac{1}{4}q^{\frac{1}{4}})}{\Gamma(\frac{1}{4})}
\end{aligned}
$$

where $\gamma(\frac{1}{4}, \frac{1}{4}q^{\frac{1}{4}})$ is the lower incomplete gamma function of $\frac{1}{4}$ evaluated at $\frac{1}{4}q^4$, and sign(q) is the sign of q.

2.5.1 STANDARD QUARTIC CDF TABLE

A tabulation of the cdf of the quartic distribution is given in Table 1:

q	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.50390	0.50780	0.51170	0.51560	0.51950	0.52340	0.52730	0.53120	0.53511
0.1	0.53901	0.54291	0.54681	0.55071	0.55461	0.55851	0.56241	0.56631	0.57021	0.57411
0.2	0.57801	0.58191	0.58580	0.58970	0.59360	0.59750	0.60139	0.60529	0.60918	0.61308
0.3	0.61697	0.62086	0.62475	0.62864	0.63253	0.63642	0.64030	0.64419	0.64807	0.65195
0.4	0.65583	0.65970	0.66357	0.66744	0.67131	0.67517	0.67903	0.68288	0.68673	0.69058
0.5	0.69442	0.69826	0.70209	0.70592	0.70974	0.71356	0.71737	0.72117	0.72497	0.72875
0.6	0.73253	0.73631	0.74007	0.74382	0.74757	0.75131	0.75503	0.75874	0.76245	0.76614
0.7	0.76982	0.77349	0.77714	0.78078	0.78441	0.78802	0.79161	0.79520	0.79876	0.80231
0.8	0.80584	0.80935	0.81284	0.81632	0.81977	0.82320	0.82662	0.83001	0.83338	0.83672
0.9	0.84005	0.84334	0.84662	0.84987	0.85309	0.85628	0.85945	0.86259	0.86570	0.86879
1.0	0.87184	0.87486	0.87785	0.88081	0.88374	0.88664	0.88950	0.89233	0.89512	0.89788
1.1	0.90060	0.90329	0.90594	0.90855	0.91113	0.91367	0.91616	0.91863	0.92105	0.92343
1.2	0.92577	0.92808	0.93034	0.93256	0.93474	0.93688	0.93898	0.94103	0.94305	0.94502
1.3	0.94695	0.94884	0.95069	0.95249	0.95426	0.95598	0.95766	0.95929	0.96089	0.96245
1.4	0.96396	0.96543	0.96686	0.96825	0.96961	0.97092	0.97219	0.97342	0.97462	0.97577
1.5	0.97689	0.97797	0.97902	0.98003	0.98100	0.98194	0.98284	0.98372	0.98455	0.98536
1.6	0.98613	0.98687	0.98759	0.98827	0.98892	0.98955	0.99015	0.99072	0.99126	0.99178
1.7	0.99228	0.99275	0.99320	0.99362	0.99403	0.99441	0.99478	0.99512	0.99545	0.99576
1.8	0.99605	0.99632	0.99658	0.99682	0.99705	0.99727	0.99747	0.99766	0.99784	0.99800
1.9	0.99816	0.99830	0.99844	0.99857	0.99868	0.99879	0.99889	0.99899	0.99907	0.99915
2.0	0.99923	0.99930	0.99936	0.99942	0.99947	0.99952	0.99957	0.99961	0.99965	0.99968
2.1	0.99971	0.99974	0.99977	0.99979	0.99981	0.99983	0.99985	0.99987	0.99988	0.99989
2.2	0.99991	0.99992	0.99993	0.99993	0.99994	0.99995	0.99995	0.99996	0.99997	0.99997
2.3	0.99997	0.99998	0.99998	0.99998	0.99998	0.99999	0.99999	0.99999	0.99999	0.99999
2.4	0.99999	0.99999	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2.5	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000

Table 1: Table of the Standard Quartic CDF

As one can see, this distribution converges must faster than the normal. For the normal, .95 of its density is obtained around 1.645 standard deviations from the mean. The Quartic obtains this same value around 1.32. This demonstrates the usefulness of the quartic as an alternative to the normal, as it has much more of its density centered closely around the mean.

2.5.2 MACLAURIN SERIES EXPANSION OF THE CDF

As was done with the normal, a MacLaurin series expansion can be used to estimate the standard quartic CDF.

Theorem 4. *For all positive values of x, the CDF for the Quartic can be given by the following summation:*

$$F(x) = \frac{1}{2} + \frac{\sqrt{2}}{\Gamma(\frac{1}{4})} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n} n!} \frac{x^{4n+1}}{4n+1}, \forall x \geq 0.$$

Proof.

$$\begin{aligned}
F(x) &= \int_{-\infty}^{x} \frac{\sqrt{2}}{\Gamma(\frac{1}{4})} \exp\left[\frac{-t^4}{4}\right] dt \\
&= \frac{1}{2} + \frac{\sqrt{2}}{\Gamma(\frac{1}{4})} \int_{0}^{x} \exp\left[\frac{-t^4}{4}\right] dt, \text{by symmetry} \\
&= \frac{1}{2} + \frac{\sqrt{2}}{\Gamma(\frac{1}{4})} \sum_{n=0}^{\infty} \int_{0}^{x} \frac{(-\frac{t^4}{4})^n}{n!} dt \\
&= \frac{1}{2} + \frac{\sqrt{2}}{\Gamma(\frac{1}{4})} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n} n!} \int_{0}^{x} t^{4n} dt \\
&= \frac{1}{2} + \frac{\sqrt{2}}{\Gamma(\frac{1}{4})} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n} n!} \frac{x^{4n+1}}{(4n+1)}
\end{aligned}$$

Now, we must show the series converges. This can be done with the Alternating Series Theorem.

$$a_n = \frac{x^{4n+1}}{(4n+1)2^{2n} n!}$$

It is clear that $a_n \geq 0$, since it is a condition placed at the formula.

It must be shown that, for large n, a_n is monotonically decreasing to apply the Alternating Series Theorem. If a_n is monotonic and decreasing, then $\frac{a_{n+1}}{a_n} \leq 1$:

$$\begin{aligned}
\frac{a_{n+1}}{a_n} &= \frac{x^{4n+5}}{2^{2n+2}(n+1)!(4n+5)} \frac{2^{2n} n!(4n+1)}{x^{4n+1}} \\
&= x^4 \frac{(4n+1)}{4(4n+1)(4n+5)} = \frac{x^4}{4(4n+5)}
\end{aligned}$$

Since this a rational function, it is clear that for any x, $\exists N_x$ such that $n \geq N$ so the quotient is less than 1. Now we must show that $a_n \to 0$ as $n \to \infty$. If we show this, then we know that $\sum_{n=0}^{\infty}(-1)^n a_n$ converges absolutely.

$$\begin{aligned} \exp\left[x^4\right] &= \sum_{n=0}^{\infty}\frac{(x^4)^n}{n!} = \sum_{n=0}^{\infty}\frac{x^{4n}}{n!} \\ &\Rightarrow \lim_{n\to\infty}\frac{x^{4n}}{n!} = 0 \\ \because 0 \leq \frac{x^{4n+1}}{(4n+1)2^{2n}n!} \leq \frac{x^{4n}}{n!} &\Rightarrow \frac{x^4}{4(4n+5)} \to 0 \text{ as } n \to \infty, \\ \text{by the Squeeze Theorem} & \end{aligned}$$

□

Thus, this can serve as a way to calculate the CDF. We can check to see if the value we obtain from the MacLaurin Series estimation aligns with the values we obtained from our first method to compute the CDF and in turn the value from the table. Let us look at $x = 1.32$. Recall the value from the table was about 0.95. Let us take a partial sum, up to $n = 20$, to see if the results from this section are valid.

$$\begin{aligned} F(1.32) &= \frac{1}{2} + \frac{\sqrt{2}}{\Gamma(\frac{1}{4})}\sum_{n=0}^{20}\frac{(-1)^n}{2^{2n}n!}\frac{1.32^{4n+1}}{4n+1} \\ &= \frac{1}{2} + (0.514968801 - 0.078171176 + 0.01648086...) = 0.5 + 0.450764221 \\ &= 0.950764221 \end{aligned}$$

which is about the value we got from the table! Using the MacLaurin Series has produced a valid way to estimate the CDF! One important thing to note is that, as mentioned, this technique only applies to positive values of x. If one wishes to find the cumulative density for a negative value of x, one needs to only exploit the symmetry of the Quartic: $F(-x) = 1 - F(x)$. This will allow one to get the density of the quartic over the entire real line.

3 ESTIMATION OF PARAMETERS

In this section, methods of estimating the parameters of the quartic distribution will be explored. We will first look at the maximum likelihood estimators for the quartic. Then, the method of moments estimators will be explored. Finally, we will develop some general properties of estimators, and look at behavior of the mean in this distribution.

3.1 MAXIMUM LIKELIHOOD ESTIMATORS

First, we will compute the maximum likelihood estimator (MLE). If the estimator is relatively simple, we can also develop hypothesis tests for μ and σ.

First, we must define the likelihood function. Let $X_1, X_2, ..., X_n$ be a simple random sample of size n following the distribution $Q(\mu, \sigma)$. Therefore, the likelihood function, $L(\mathbf{X}|\mu, \sigma)$, is:

$$\begin{aligned} L(X|\mu,\sigma) &= f(X_1)f(X_2)...f(X_n) \\ &= \left[\frac{\sqrt{2}}{\Gamma(1/4)\sigma}\right]^n \exp\left[-\frac{1}{4}\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^4\right]. \end{aligned}$$

Next, we will use the following substitution to simply the computations:

$$Q(x|\mu,\sigma) = \sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^4$$

If we expand $(x_i - \mu)^4$, we obtain:

$$\begin{aligned} &\frac{n}{\sigma^4}\sum_{i=1}^{n}\left(x_i^4 - 4x_i^3\mu + 6x_i^2\mu^2 - 4x_i\mu^3 + \mu^4\right) \\ &= \frac{n}{\sigma^4}\left(M_4 - 4M_3\mu + 6M_2\mu^2 - 4M_1\mu^3 + n\mu^4\right), \end{aligned}$$

where $M_j = \frac{1}{n}\sum_{i=1}^{n} x_n^j$, or the j^{th} sample moment.

We can take the natural logarithm of the likelihood function in order to simplify

computations. The log-likelihood, $\mathcal{L}(\mathbf{X}|\mu,\sigma)$ is:

$$\mathcal{L}(\mu,\sigma|\boldsymbol{x}) = n\ln\left(\sqrt{2}\right) - n\ln\left(\Gamma(1/4)\right) - n\ln(\sigma) - \frac{1}{4}Q(x|\mu,\sigma).$$

In order to acquire the estimators for μ and σ, we must maximize $\mathcal{L}$ with respect to the parameters, which will involve finding the partial derivatives of $\mathcal{L}$, and setting them to zero.

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\mu} &= 0 \Leftrightarrow \frac{\partial Q}{\partial\mu} = 0 \\
\frac{\partial\mathcal{L}}{\partial\sigma} &= \frac{-n}{\sigma} - \frac{1}{4}\frac{\partial Q}{\partial\sigma} \\
\frac{\partial^2\mathcal{L}}{\partial\mu^2} &= \frac{1}{4}\frac{\partial^2 Q}{\partial\mu^2} \\
\frac{\partial^2\mathcal{L}}{\partial\mu\partial\sigma} &= \frac{1}{4}\frac{\partial^2 Q}{\partial\mu\partial\sigma} \\
\frac{\partial^2\mathcal{L}}{\partial\sigma^2} &= \frac{n}{\sigma^2} - \frac{1}{4}\frac{\partial^2 Q}{\partial\sigma^2}
\end{aligned}$$

So now, let us find the partial derivatives of Q in order to solve the system of equations:

$$\begin{aligned}
\frac{\partial Q}{\partial\sigma} &= -\frac{4n}{\sigma^5}\left\{M_4 - 4M_3\mu + 6M_2\mu^2 - 4M_1\mu^3 + \mu^4\right\}, \\
\frac{\partial^2 Q}{\partial\sigma^2} &= \frac{20n}{\sigma^6}\left\{M_4 - 4M_3\mu + 6M_2\mu^2 - 4M_1\mu^3 + \mu^4\right\}, \\
\frac{\partial Q}{\partial\mu} &= \frac{4n}{\sigma^4}\left\{M_3 - 3M_2\mu + 3M_1\mu^2 - \mu^3\right\}, \\
\frac{\partial^2 Q}{\partial\mu^2} &= -\frac{12n}{\sigma^4}\left\{-M_2 + 2M_1\mu - \mu^2\right\}, \\
\frac{\partial^2 Q}{\partial\mu\partial\sigma} &= -\frac{16n}{\sigma^5}\left\{M_3 - 3M_2\mu + 3M_1\mu^2 - \mu^3\right\},
\end{aligned}$$

Thus, we set the partial with respect to μ to zero, and solve for the parameter. This results in a cubic equation. Solving it yields:

$$\hat{\mu} = M_1 + \frac{2^{1/3}\left(-9M_1^2 + 9M_2\right)}{9\left(-2M_1^3 + 3M_1M_2 - M_3 + \sqrt{-3M_1^2M_2^2 + 4M_2^3 + 4M_1^3M_3 - 6M_1M_2M_3 + M_3^2}\right)^{1/3}} - \frac{\left(-2M_1^3 + 3M_1M_2 - M_3 + \sqrt{-3M_1^2M_2^2 + 4M_2^3 + 4M_1^3M_3 - 6M_1M_2M_3 + M_3^2}\right)^{1/3}}{2^{1/3}}$$

The two imaginary roots were discarded. Let

$$r = \sqrt{-3M_1^2M_2^2 + 4M_2^3 + 4M_1^3M_3 - 6M_1M_2M_3 + M_3^2}.$$

This will simplify the expression as follows:

$$M_1 + \frac{2^{1/3}\left(-9M_1^2 + 9M_2\right)}{9\left(-2M_1^3 + 3M_1M_2 - M_3 + \sqrt{r}\right)^{1/3}} - \frac{\left(-2M_1^3 + 3M_1M_2 - M_3 + \sqrt{r}\right)^{1/3}}{2^{1/3}}$$

Now let us consider estimation of σ. If

$$\frac{n}{\sigma} = -\frac{4n}{\sigma^5}\left\{M_4 - 4M_3\mu + 6M_2\mu^2 - 4M_1\mu^3 + \mu^4\right\},$$

then

$$\hat{\sigma^4} = M_4 - 4M_3\mu + 6M_2\mu^2 - 4M_1\mu^3 + \mu^4.$$

However,

$$\frac{\partial\mathcal{L}}{\partial\mu} = 0 \Rightarrow -S_3\mu + 3M_2\mu^2 + -3M_1\mu^3 + \mu^4 \Rightarrow \mu^4 = M_3\mu - 3M_2\mu^2 + 3M_1\mu^3$$

by simply multiplying the partial of $\mathcal{L}$ with respect to μ through by μ. So, we substitute the value of μ^4 into the equation for $\hat{\sigma^4}$ since the equations must be solved simultaneously. Doing so, we acquire:

$$\hat{\sigma^4} = M_4 - M_3\hat{\mu} + 3M_2\hat{\mu^2} - M_1\hat{\mu^3}$$

as the maximum likelihood estimator of σ. This equation can be solved numerically.

3.2 METHOD OF MOMENTS ESTIMATORS

The method of moments estimators are acquired by setting the sample moments to the population moments, and solving. It is a relatively simple method that should produce clean estimators, unlike the MLEs.

The method of moments are acquired by setting the sample moments equal to the population moments, and solving, This method should produce cleaner estimates than the ones previously derived. We shall use the first and second central population moments, denoted by μ_1 and μ_2 respectively, and set them equal to their sample counterparts, denoted by M_1 and M_2 respectively.

$$\begin{aligned}
M_1 &= E(\mu + Q\sigma) = \hat{\mu} + E(Q)\hat{\sigma} = \hat{\mu} \\
M_2 &= E[(\mu + Q\sigma)^2] \\
&= E(\mu^2 + 2Q\sigma\mu + Q^2\sigma^2) \\
&= \mu^2 + E(Q^2)\sigma \\
&= M_1^2 + \frac{2\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}\sigma^2 \\
\Rightarrow \quad \hat{\sigma^2} &= \frac{\Gamma(\frac{1}{4})(M_2 - M_1^2)}{2\Gamma(\frac{3}{4})}
\end{aligned}$$

Though these moments are much simpler in appearance, they may not be unbiased, minimum-variance estimators. A third method of estimation shall be explored in hopes that it can produce estimators with more favorable properties than the method of moments estimators.

3.3 FINAL THOUGHTS ON ESTIMATION

In this section, two methods of estimation were explored. The MLEs are generally very nice estimators, but their goodness as estimators cannot be verified due to their complexity. The method of moments, on the other hand, was simpler in appearance and application. However, as shown in the section exploring a minimum-variance, unbiased estimator, the result it produced as an estimate for μ is rather poor. Unfortunately, there seems to be no good solution for estimating the mean of the quartic distribution. The author would recommend the method of moment estimators, due to their relative ease to work with, but would like to stress that they most likely do not have good properties. Estimation may be a good place for further research on the quartic, and may provide an interesting problem for a more advanced researcher than the author.

4 GENERALIZED QUARTIC DISTRIBUTION

Given that the Quartic was developed by taking the exponentiated term of the Normal to the fourth power instead of the second, it stands to reason that we can generalize this to any even power. Thus, we shall define a new probability distribution called the Generalized Quartic, as follows:

A random variable, X, is said to be distributed according to the Generalized Quartic distribution if its probability density function, pdf, is of the form:

$$f(x|\mu,\sigma,m) = (2^{\frac{1}{2m}} m^{\frac{1-2m}{2m}} \Gamma(1/2m)\sigma)^{-1} \exp\left(-\frac{(x-\mu)^{2m}}{2m\sigma^{2m}}\right) \tag{1}$$

where μ is the location parameter, σ is the scale parameter, and $m \in N$ is a shape parameter.

One can see that both the normal and quartic are particular instances of this generalized distribution, when $m = 1$ and $m = 2$ respectively. Again, like the normal and quartic, we can refer to a standard version of this distribution, where $\mu = 0$ and $\sigma = 1$. Some properties

of this distribution will be briefly explored. This, in turn, will also shed more light on the Quartic.

4.1 DISTRIBUTION OF STATISTICS

We are able to derive the distributions of various statistics for the generalized quartic. This will give the distribution for these statistics in not only the quartic and normal, but an entire class of distributions.

Theorem 5. *The statistics X^{2m} and $\sum_{i=1}^{n} X_i^{2m}$ are distributed as follows:*

$$X^{2m} \sim Gamma\left(\frac{1}{2m}, 1\right)$$

$$\sum_{i=1}^{n} X_i^{2m} \sim Gamma\left(\frac{1}{2m}, n\right)$$

Proof.

$$f(X) = \frac{\exp\left(-\frac{1}{2m}(\frac{X}{\sigma})^{2m}\right)}{2^{\frac{1}{2m}} m^{\frac{1-2m}{2m}} \Gamma(1/2m)\sigma}$$

Substitute

$$Y = X^{2m} \Rightarrow X = Y^{\frac{1}{2m}}$$
$$J = \frac{dx}{dy} = \frac{1}{2m} Y^{\frac{1}{2m}-1}$$

Yielding:

$$\begin{aligned} f(Y) &= \frac{\exp\left(-\frac{Y}{2m\sigma^{2m}}\right) Y^{\frac{1}{2m}-1}}{2m 2^{\frac{1}{2m}} m^{\frac{1-2m}{2m}} \Gamma(1/2m)\sigma} \\ &= \frac{\exp\left(-\frac{Y}{2m\sigma^{2m}}\right) Y^{\frac{1}{2m}-1}}{2^{\frac{1}{2m}+1} m^{\frac{1}{2m}} \Gamma(1/2m)\sigma} \end{aligned}$$

Substitute again:

$$\frac{u}{2} = \frac{Y}{2m\sigma^{2m}} \Leftrightarrow Y = um\sigma^{2m}$$
$$J = \frac{dy}{du} = m\sigma^{2m}$$

Yielding:

$$\begin{aligned}
f(u) &= \frac{\exp\left(-\frac{u}{2}\right)(um\sigma^{2m})^{\frac{1}{2m}-1}}{2m2^{\frac{1}{2m}}m^{\frac{1-2m}{2m}}\Gamma(1/2m)\sigma} m\sigma^{2m} \\
&= \left(\frac{1}{2}\right)\frac{\exp -\frac{u}{2}u^{\frac{1}{2m}-1}}{2^{\frac{1}{2m}}\Gamma(1/2m)} \\
\therefore Y &\sim \left(\frac{1}{2}\right)Gamma\left(\frac{1}{2m},2\right) \\
&= Gamma\left(\frac{1}{2m},1\right), \text{By the scalar property of the Gamma distribution} \\
&\Rightarrow \sum_{i=1}^{n} Y_i^{2m} \sim Gamma\left(\frac{1}{2m},n\right)
\end{aligned}$$

□

5 CONCLUDING REMARKS

This paper has laid out the basic mathematical and statistical properties of a new distribution. The relative complexity of the distribution has resulted in some of the tasks being difficult, such as finding a closed-form solution to the moment generating function or MLEs that do not need to be numerically approximated. Despite that, solutions to these problems were found, and the author is quite pleased with the overall results in the paper.

Hopefully, the results in this paper will be useful should anyone like to continue research on this distribution. Estimation could prove to be a quite interesting problem regarding the quartic, as discussed earlier. Furthermore, there are many interesting problems regarding the generalized form of the quartic, such as developing estimators for that class

of distributions or developing its mathematical properties, much like what was done in this paper. Lastly, there was little discussion of the application of the quartic. Due to its unique shape, the quartic could potentially prove to be a useful distribution in application. The results developed in this paper could be important for anyone wishing to explore those applications.

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